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Paramagnetic effect of light quark loops on chiral symmetry breaking*

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Abstract: We argue that light quark loops produce a paramagnetic suppression of infrared-sensitive order parameters such as $\langle \bar{q}q \rangle$, as the number N_f of light fermions increases. The possibly strong dependence of $\langle \bar{q}q \rangle$ on N_f is related to the observed Zweig rule violation in the scalar channel. Presuming the existence of a chiral phase transition for not too large N_f, we discuss the phenomenological possibilities of separately determining the two-flavour and three-flavour condensates and the quark mass ratio $r = 2m_s/(m_u + m_d)$. The issue is closely related to the interpretation of new forthcoming precise $\pi\pi$ data at low energy.

Keywords: Spontaneous Symmetry Breaking, 1/N Expansion, QCD, Chiral Lagrangians.

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Contents

1. Introduction

The purpose of this paper is to reconsider our understanding of the mechanism of spontaneous chiral symmetry breaking (SχSB) in QCD as reflected by the (possibly small) size of $\bar{q}q$ condensates and to interpret its relation to the Zweig rule (ZR) violation in the scalar channel and to the quark mass ratio $r = 2m_s/(m_u + m_d)$. New precise experimental results which are expected to come soon [1, 2, 3] will merely concern low-energy $\pi\pi$ scattering which is governed by the chiral dynamics of u- and d-quarks with the s-quark playing at most the role of a "sea-side spectator". From this point of view the existence of a relationship between low-energy $\pi\pi$ observables and the quark mass ratio $r = m_s/m$ ($m = m_u = m_d$) is by no means obvious. A related question concerns determinations of SU(2)$\times$SU(2) low-energy constants from SU(3)$\times$SU(3) observables: for example, the low energy constants l_4 and l_3, which are essential to assess the prediction of standard chiral perturbation theory (SχPT) [4] for $\pi\pi$ s-wave scattering lengths [5], are usually inferred using the experimental values of F_K/F_π and of the K- and η-meson masses respectively.

The theoretical counterpart of these and similar phenomenological questions concerns the dependence of various order parameters of SχSB on the number N_f of massless quark flavours. This N_f-dependence is an effect of light quark loops and it is generally expected to be rather weak: it is suppressed in the large-N_c limit and it violates the Zweig rule which are both considered as a good approximation to the real QCD (with a possible exception of the anomalous 0^{-+} channel). On the other hand, more recent investigations suggest that N_f-dependent light-quark loop effects could sometimes be rather important. First, Nature does not seem to respect the large-N_c predictions in the scalar channel which is not dominated by ideally mixed $\bar{q}q$ nonets, as required by the Zweig rule (see ref. [6] for a review and ref. [7] for further references). Next, some recent lattice simulations with dynamical fermions [8, 9] observe a rather strong N_f-dependence of SχSB signals (see also ref. [10]). Finally, a new method of estimating the variation of the chiral condensate between $N_f = 2$ and $N_f = 3$ from the data has recently been proposed [11]. Using as input experimental informations on ZR violation in the scalar channel at low and medium energies, a large variation of $\langle \bar{q}q \rangle$ has been found.

For sufficiently large N_f/N_c, the existence of chiral phase transitions is generally expected on the basis of the behaviour of the perturbative QCD β-function [12]. Various approaches have been proposed to study the transitions arising when N_f increases. Investigations about the QCD conformal window [12], where the theory is asymptotically free in the ultraviolet, but is governed by a non-trivial fixed point in the infrared, suggest a restoration of chiral symmetry for $N_f \sim 10$ at $N_c = 3$ [13]. The analysis of gap equations in the same conformal window puts a slightly different bound, with a critical N_f above 12 [14]. On the other hand, the instanton liquid model indicates that instantons do not significantly contribute to Chiral Symmetry Breaking for $N_f > 6$ [15].

We would like to stress that particular properties of vector-like gauge theories such as QCD allow a new type of non-perturbative interpretation of SχSB and of its N_f-dependence, suggesting a natural link between a suppression of $\langle \bar{q}q \rangle$ and an enhancement of ZR violation in the scalar channel for not too large N_f/N_c. No reference is made to concepts and methods such as gap equation, mean field approximation, or the critical effective coupling, which in confining non-abelian gauge theories lack a truly non-perturbative gauge-invariant definition. Instead, QCD may be formulated in euclidean space-time and the mechanism of SχSB can be related to the dynamics of the lowest modes of the hermitean Dirac operator [16, 17, 18]

$$H[G] = \gamma_\mu(\partial_\mu + iG_\mu) = H^\dagger[G]\,, \tag{1.1}$$

averaged over gauge field configurations $G_\mu(x)$. In particular, the role of increasing N_f naturally appears as a consequence of the paramagnetic effect of light quark loops: it tends to suppress the order parameters that are dominated by the small eigenvalues of H. Whether this suppression is strong enough to imply a phase transition for not too large N_f is a dynamical question which at present we are not able to answer analytically. At least one can arrive at a theoretically coherent framework suggesting a semi-quantitative understanding of various aspects of SχSB and of their possible interconnection.

2. N_f-dependence of infrared-dominated order parameters

The theory will be considered in a four-dimensional euclidean box $L \times L \times L \times L$ with (anti)periodic boundary conditions modulo a gauge transformation (i.e. on a torus). We are interested in gauge invariant correlation functions which break the chiral symmetry of the vacuum, i.e. in vacuum expectations of operators which do not contain the singlet representation of the chiral symmetry group. All external momenta are set to 0 and all quarks are first taken to be massive. At the end we consider the limit in which the first N_f lightest quarks become massless keeping the remaining quark masses fixed. One randomly chooses a gauge field configuration $G_\mu(x)$ and performs the Grassmann integral over quark fields. The result can be formally expressed in terms of eigenvalues and eigenfunctions of the hermitean Dirac operator (1.1):

$$H[G]\phi_n = \lambda_n[G]\phi_n\,, \qquad \int dx\, \phi_n^\dagger(x)\phi_k(x) = \delta_{nk}\,. \tag{2.1}$$

Since $H[G]$ anticommutes with $\gamma_5 = \gamma_5^\dagger$, the spectrum is symmetric around 0. In addition, for gauge field configurations with non-vanishing winding number $\nu \in Z$, the spectrum contains $|\nu|$ topological zero modes. Positive eigenvalues will be ranged in the ascending order and numerated by a positive integer, defining $\lambda_{-n} = -\lambda_n$. Notice that the eigenvalues λ_n and the wave functions $\phi_n(x)$ are entirely given by the gauge field configuration $G_\mu(x)$; in particular, they are independent of the quark flavour and of the quark masses.[1] The dependence on quark flavour and masses arises from the propagator

$$S_j(x,y|G) = \sum_n \frac{\phi_n(x)\phi_n^\dagger(y)}{m_j - i\lambda_n[G]} \tag{2.2}$$

and from the fermionic determinant $\det(M - iH) = \prod_j \Delta(m_j|G)$, where the single flavour determinant can be expressed as

$$\Delta(m|G) = m^{|\nu|} \prod_{n>0} \frac{m^2 + \lambda_n^2[G]}{m^2 + \omega_n^2}\,. \tag{2.3}$$

The G-independent numbers ω_n, which essentially coincide with the free eigenvalues, provide a convenient overall normalization of (2.3) and will be specified shortly. An integral over fermion fields of any product of bilinear quark currents can be expressed in terms of the propagator (2.2) and of the determinant (2.3). The next step then consists in taking an average over all gluon configurations. Before we comment on this last point, let us concentrate on the simplest example of an order parameter of SχSB.

The chiral condensate $\langle \bar{u}u\rangle$, where u stands for the lightest quark, will be considered in the SU(N_f) $\times$ SU(N_f) chiral limit

$$m_1 = m_2 = \cdots = m_{N_f} = m \to 0\,, \qquad m_j \neq 0\,, \quad j > N_f \tag{2.4}$$

[1]The wave functions ϕ_n live in the spin $\times$ colour space and they transform as the fundamental representation of SU(N_c) and as a 4-dimensional O(4) spinor respectively.

and it will be denoted as $-\Sigma(N_f)$. One has

$$\Sigma(N_f) = \lim \frac{1}{L^4} \left\langle\!\!\left\langle \int dx \mathrm{Tr}\ S(x,x|G) \right\rangle\!\!\right\rangle_{N_f} = \lim \frac{1}{L^4} \left\langle\!\!\left\langle \sum_n \frac{m}{m^2+\lambda_n^2} \right\rangle\!\!\right\rangle_{N_f} . \quad (2.5)$$

Hereafter, lim denotes the $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ limit (2.4) preceeded by the large volume limit. The symbol $\langle\!\langle\ \rangle\!\rangle_{N_f}$ represents the normalized ($\langle\!\langle 1\rangle\!\rangle_{N_f} = 1$) average over gauge-field configurations weighted by the fermionic determinant,

$$\langle\!\langle \Gamma \rangle\!\rangle_{N_f} = Z^{-1} \int d\mu[G]\ \Gamma\ \Delta^{N_f}(m|G) \prod_{j>N_f} \Delta(m_j|G) \exp\{-S[G]\}, \quad (2.6)$$

where $S[G]$ stands for the Yang-Mills action. Since every gauge-field configuration $G_\mu(x)$ can be globally characterized by the corresponding set of Dirac eigenvalues λ and eigenvectors ϕ, the functional integral (2.6) may be viewed as an average over all possible Dirac spectra. The probability distribution of Dirac eigenvalues should be, in principle, calculable from the theory itself. In practice, it requires a non-perturbative regularization and renormalization of the gluonic average (2.6) which (as in perturbation theory) may depend on the observable Γ. Whilst this problem is hard to solve in general, in the particular case of chiral order parameters such as (2.5), the formal structure of eq. (2.6) suggests some possibly interesting properties of SχSB even before an analytic solution becomes available.

The fact that $\langle \bar{u}u \rangle$ is an order parameter of SχSB is reflected by the vanishing of the $m \to 0$ limit of eq. (2.5) taken at finite volume. In order to get a non-trivial result, the large volume limit has to be taken first and the spectrum of the Dirac operator has to become sufficiently dense around the origin. Actually, the average distribution of the smallest Dirac eigenvalues is all what matters: if in eq. (2.5) one cuts the infrared end of the spectrum and sticks to $|\lambda_n| > \Lambda$, the result would be zero, no matter how small Λ is. For the same reason, the ultraviolet divergences of the sum $\sum_n$ in eq. (2.5) become irrelevant in the chiral limit. It turns out that only eigenvalues which in the average accumulate like $1/L^4$ contribute to the chiral condensate $\langle \bar{q}q \rangle$ [16, 17, 18]. A similar discussion applies to other order parameters of SχSB, in particular to the coupling F_π of Goldstone bosons to the axial current, whose square can be expressed as the two-point correlator of left-handed and right-handed currents $\langle L_\mu R_\nu \rangle$ at zero momentum transfer. Denoting by $F^2(N_f)$ the $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ chiral limit (2.4) of F_π^2, one has [19]

$$F^2(N_f) = \lim \frac{1}{L^4} \left\langle\!\!\left\langle \sum_{k,n} \frac{m}{m^2+\lambda_k^2} \frac{m}{m^2+\lambda_n^2} J_{kn} \right\rangle\!\!\right\rangle_{N_f} , \quad (2.7)$$

where lim has the same meaning as in eq. (2.5) and

$$J_{kn} = \frac{1}{4} \sum_\mu \left| \int dx\ \phi_k^\dagger(x) \gamma_\mu \phi_n(x) \right|^2 . \quad (2.8)$$

Due to the Goldstone theorem, $F^2(N_f) \neq 0$ is both sufficient and necessary for the chiral symmetry $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ to be spontaneously broken. Again, $F^2(N_f)$ merely receives contributions from the lowest Dirac eigenvalues but it is less infrared sensitive than $\langle \bar{u}u \rangle$: the eigenvalues behaving in the average as $1/L^2$ could now be sufficient to produce a non-zero value of (2.7), since there are two factors $m/(m^2+\lambda^2)$ for a single inverse power of volume [19].

Eq. (2.6) suggests that for $m \to 0$ the effect of the fermionic determinant and the N_f-dependence will be stronger for observables Γ which are dominated by the lowest Dirac eigenvalues. This observation follows from the rigorously proven inequality [17]

$$|\lambda_n[G]| < C \frac{n^{1/d}}{L} \equiv \omega_n \,, \tag{2.9}$$

where d is the space-time dimension ($d = 4$ in our case) and C is a constant independent of the gauge field configuration $G_\mu(x)$, of the integer n and of the volume $V = L^d$. (In general, C depends on the shape of the space time manifold, once the volume has been fixed.) The existence of the uniform upper bound (2.9) reflects the paramagnetic response of the euclidean Dirac spectrum to an external gauge field [20]. It allows to split the single flavour determinant (2.3) into infrared and ultraviolet parts [21]: one chooses a cutoff Λ and defines an integer K such that $\omega_K = \Lambda$. The determinant is then written

$$\Delta(m|G) = m^{|\nu|} \Delta_{\mathrm{IR}}(m|G) \Delta_{\mathrm{UV}}(m|G) \,, \tag{2.10}$$

where Δ_{IR} involves the first K non-zero eigenvalues and is bounded by 1 as a consequence of the inequality (2.9),

$$\Delta_{\mathrm{IR}}(m|G) = \prod_{k=1}^{K} \frac{m^2 + \lambda_k^2[G]}{m^2 + \omega_k^2} < 1 \,. \tag{2.11}$$

One may expect that in the case of chiral order parameters such as $\langle \bar{u}u \rangle$ (2.5) or F^2 (2.7) which are entirely dominated by the infrared extremity of the Dirac spectrum, the Δ_{IR} part of the determinant will describe the bulk of the effect of light quark loops in eq. (2.6). This effect should be paramagnetic,

$$\Sigma(N_f + 1) < \Sigma(N_f), \qquad F^2(N_f + 1) < F^2(N_f) \,, \tag{2.12}$$

indicating the suppression of chiral order parameters with increasing N_f. How strong is this suppression depends on how sensitive is the observable Γ in eq. (2.6) to the smallest Dirac eigenvalues for large volumes and small quark masses. For this reason one may expect a stronger suppression in the case of $\langle \bar{q}q \rangle$ than for F_π. On the other hand, for observables that are not dominated by the infrared extremity of the Dirac spectrum, the sensitivity to the determinant and to N_f can remain marginal, as expected in the large-N_c limit.

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3. Why the scalar channel does not obey large-N_c predictions

We now turn to the connection between flavour dependence of order parameters of SχSB and the observed violation of the Zweig rule in the scalar channel. As before, we consider first N_f light flavors of common mass $m \to 0$ and denote by s the (N_f+1)-th quark, whose mass m_s is non-zero, but still considered as light compared to the scale of the theory. For $N_f = 2$, this corresponds to the situation in real QCD. $\Sigma(N_f)$ is a function of m_s, and its derivative reads

$$\begin{aligned}\frac{\partial}{\partial m_s}\Sigma(N_f) &= \lim_{m\to 0}\int dx \langle \bar{u}u(x)\bar{s}s(0)\rangle^c \equiv \Pi_Z(m_s) \\ &= \lim \frac{1}{L^4}\left\langle\!\!\left\langle \left(\sum_k \frac{m}{m^2+\lambda_k^2}\right)\left(\sum_n \frac{m_s}{m_s^2+\lambda_n^2}\right)\right\rangle\!\!\right\rangle^c_{N_f}, \qquad (3.1)\end{aligned}$$

where the notations are as before and the superscript c stands for the connected part. Since $\Sigma(N_f) \to \Sigma(N_f+1)$ for $m_s \to 0$, one can write

$$\Sigma(N_f) = \Sigma(N_f+1) + \int_0^{m_s} d\mu\, \Pi_Z(\mu) = \Sigma(N_f+1) + m_s Z^S_{\text{eff}}(m_s) + O(m_s^2 \log m_s)\,. \quad (3.2)$$

In general, and for $N_f \sim 2-3$, the difference $\Delta(N_f) = \Sigma(N_f) - \Sigma(N_f+1)$ is expected to be negligible compared to $\Sigma(N_f)$ for two independent reasons: first, it is chirally suppressed due to the smallness of m_s relative to the condensate $\Sigma(N_f+1)$, at least provided the latter is of the standard "normal" size. Second, the connected correlator (3.1) of scalar quark densities of different flavours is suppressed in the large-N_c limit relative to $\Sigma(N_f)$. An important N_f-dependence of the condensate would imply that *both* these arguments fail. We argue that this should naturally be expected close to the critical point $n_{\text{crit}}(N_c)$ at which $\Sigma(N_f)$ vanishes. Suppose that for a given value of N_c (e.g. $N_c = 3$), we have $N_f + 1 < n_{\text{crit}}(N_c)$, so that $\Sigma(N_f+1)$ is still non-zero but already small. Then (for the actual value of m_s) the condensate term need not dominate the expansion (3.2) in powers of m_s, not because the chiral expansion breaks down but due to the suppression of $\Sigma(N_f+1)$. The generalized chiral perturbation theory (GχPT) is precisely designed to cope with such a situation [22, 23, 24]. The suppression of the condensate means that near the critical point, the average density of states [16]

$$\frac{1}{L^4}\langle\!\langle \rho(\lambda)\rangle\!\rangle_{N_f}\,, \qquad \rho(\lambda) = \sum_n \delta(\lambda - \lambda_n[G])\,, \qquad (3.3)$$

drops for $\lambda \sim m$. Indeed, eq. (2.5) can be rewritten as

$$\Sigma(N_f) = 2\lim \int_0^\infty \frac{du}{1+u^2}\frac{1}{L^4}\langle\!\langle \rho(mu)\rangle\!\rangle_{N_f}\,. \qquad (3.4)$$

It is then natural to expect that the proximity of a phase transition will further manifest itself by an increase of fluctuations of the density of states and/or by an enhancement of the density-density correlation $L^{-4}\langle\langle\rho(\lambda)\rho(\lambda')\rangle\rangle^c_{N_f}$ for $\lambda \sim \lambda' \sim m$. This quantity determines the variation of the condensate, see eqs. (3.1) and (3.2):

$$\frac{\partial}{\partial m_s}\Sigma(N_f) = 4\lim \int_0^\infty \frac{du}{1+u^2}\frac{dv}{1+v^2}\frac{1}{L^4}\langle\langle\rho(mu)\rho(m_s v)\rangle\rangle^c_{N_f}. \tag{3.5}$$

Far away from the critical point, the correlation of small Dirac eigenvalues should not be important, as predicted by the large-N_c limit. Since this limit leads to a suppression of quark loops, any infrared-dominated order parameter becomes independent of N_f. Therefore, the large-N_c limit prevents Σ from vanishing, since the critical number of flavours $n_{\rm crit}(N_c)$ moves away to infinity as $N_c \to \infty$. This asymptotic behaviour is also supported by perturbative calculations, in which N_f and N_c usually arise through their ratio. Large-N_c expansion is therefore expected to converge slowly for N_f fixed just below the critical point $n_{\rm crit}(N_c)$, and to yield irreparably false results for N_f above it. This argument merely concerns the vacuum channel 0^{++}. The variation of any chiral order parameter between N_f and N_f+1 is given by a correlation function which violates the Zweig rule precisely in that channel, cf eq. (3.2): a strong variation would imply the existence of $J^{PC} = 0^{++}$, $\mathrm{SU}_V(N_f)$-singlet states strongly coupled both to the first N_f light quarks and to the *scalar* density $\bar{s}s$ of the (N_f+1)-th quark. The proximity of a phase transition could then explain in a natural way why the spectrum of 0^{++} states is not dominated by ideally mixed scalar mesons, presenting significant discrepancies with large-N_c predictions.[2]

In the following we concentrate on the actual case $N_f = 2-3$, having in mind the possibility that (for $N_c = 3$) the real world might already be close to the critical point. It has recently been pointed out [11] that the correlation function (3.1) satisfies a well convergent sum rule which allows a phenomenological estimate of the N_f-dependence of the condensate between $N_f = 2$ and $N_f = 3$,

$$\Pi_Z(m_s) = \frac{1}{\pi}\int_0^\infty \frac{dt}{t}\,\sigma(t)\,, \tag{3.6}$$

where the spectral function $\sigma(p^2)$ (defined in Minkowski space-time) collects ZR violating contributions which couple both to $\bar{u}u$ and $\bar{s}s$

$$\sigma(p^2) = \frac{1}{2}\sum_n (2\pi)^4\delta^{(4)}(p-P_n)\langle 0|\bar{u}u|n\rangle\langle n|\bar{s}s|0\rangle\,. \tag{3.7}$$

Since we take $m_u = m_d = m$ and isospin symmetry cannot be spontaneously broken [26], only isoscalar states $|n\rangle$ contribute in eq. (3.7) and $\langle 0|\bar{u}u|n\rangle = \langle 0|\bar{d}d|n\rangle$. Under the plausible assumption that the dominant contribution comes from two-

[2] Connections with scenarios invoking trace anomaly and light dilatons [25] remain to be seen. We thank H. Leutwyler for bringing our attention to this question.

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particle states $(|\pi\pi\rangle, |K\bar{K}\rangle, \ldots)$, the spectral function (3.7) can be reconstructed from the corresponding multi-channel T-matrix, which contains the experimental information on the size of Zweig rule violation in the 0^{++} channel, such as the effect of the $f_0(980)$ resonance. Using further χPT to normalize the solution of the multi-channel Muskhelishvili-Omnès equation, one can obtain the corresponding form factors $\langle 0|\bar{q}q|n\rangle$. Within the standard version of χPT, a strong N_f-dependence of the condensate has been found [11]: $\Sigma(3)/\Sigma(2) = 1 - 0.54 \pm 0.27$. We shall further comment on this result shortly.

4. χPT considerations

The case of a large Zweig-rule violation leading to a substantial difference between the $N_f = 2$ and $N_f = 3$ condensates has never been fully included into the χPT analysis before. This leads us to carefully reconsider the GχPT relation between the actual size of the condensate(s), the quark mass ratio $r = m_s/m$, and some low energy observables. What matters is the renormalization group invariant product $m\Sigma$ in physical units of $F_\pi^2 M_\pi^2$, i.e. the Gell-Mann-Oakes-Renner ratio(s) [27]

$$X(N_f) = \frac{2m\Sigma(N_f)}{F_\pi^2 M_\pi^2}\,. \tag{4.1}$$

The standard chiral expansion presumes that both $X(2)$ and $X(3)$ are close to 1 which is, a priori, hardly compatible with an important flavour dependence of the condensate. On the other hand, in GχPT, the GOR ratio (4.1) can be significantly below 1, leaving enough space for a large N_f-dependence. It is worth stressing that an important ZR violation would not affect the quantitative relation between $X(2)$ and low energy $\pi\pi$ observables provided it is systematically based on $SU(2) \times SU(2)$ GχPT. The reason is not a "bad convergence" of the expansion in the strange quark mass. The ZR violation effects do not manifest themselves through a specific low-energy constants in the $N_f = 2$ effective lagrangean, whereas in the three-flavour $\mathcal{L}_{\text{eff}}$ they show up as extra (ZR violating) low energy constants $L_6(\mu)$ and $L_4(\mu)$ which are a priori unknown and have never been determined experimentally.[3] Hence, $X(2)$ remains accessible to experiment via low-energy $\pi\pi$ phases [23, 24], azimuthal asymmetries in the decay $\tau \to 3\pi\nu_\tau$ [28], or in the reaction $\gamma + \gamma \to 3\pi$ [29]. The question remains how this information is related to the size of $X(3)$, to the ZR violation in the 0^{++} channel and to the quark mass ratio $r = m_s/m$.

A partial answer to these questions can be obtained from a new look at old $SU(3) \times SU(3)$ GχPT expansions of kaon and pion masses (see ref. [22] and ref. [24,

[3]The reason of this difference resides in group theory: while the ZR violating correlator $\langle \bar{u}u(x)\bar{d}d(0)\rangle$ does not break the chiral symmetry $SU(2)\times SU(2)$, the two-point function $\langle \bar{u}u(x)\bar{s}s(0)\rangle$ is an order parameter for both $N_f = 2$ and $N_f = 3$ chiral symmetry.

appendix A]), rewritten as

$$F_\pi^2 M_\pi^2 = 2m\left[\Sigma(3) + (m_s + 2m) Z_{\rm eff}^S(m_s)\right] + 4m^2 A_{\rm eff} + F_\pi^2 \delta M_\pi^2, \tag{4.2}$$

$$\begin{aligned} F_K^2 M_K^2 = {} & (m + m_s)\left[\Sigma(3) + (m_s + 2m) Z_{\rm eff}^S(m_s)\right] + \\ & + (m + m_s)^2 A_{\rm eff} + F_K^2 \delta M_K^2. \end{aligned} \tag{4.3}$$

These formulae collect in a scale independent manner all contributions linear *and* quadratic in quark masses. They are useful to the extent that $\delta M_P^2 \ll M_P^2$, which is certainly a much weaker requirement than the assumption which is at the basis of SχPT. The constants $Z_{\rm eff}^S$ and $A_{\rm eff}$ are related to the quark-mass independent constants of the $O(p^2)$ GχPT lagrangean, Z_0^S and A_0, renormalized at a scale μ:

$$Z_{\rm eff}^S(m_s) = 2F^2(3) Z_0^S(\mu) - \frac{B_0^2}{16\pi^2}\left\{\log\frac{\bar{M}_K^2}{\mu^2} + \frac{2}{9}\log\frac{\bar{M}_\eta^2}{\mu^2}\right\}, \tag{4.4}$$

$$A_{\rm eff} = F^2(3) A_0(\mu) - \frac{B_0^2}{32\pi^2}\left\{\log\frac{\bar{M}_K^2}{\mu^2} + \frac{2}{3}\log\frac{\bar{M}_\eta^2}{\mu^2} + 3\log\frac{M_K^2}{\bar{M}_K^2} + \frac{10}{9}\log\frac{M_\eta^2}{\bar{M}_\eta^2}\right\}, \tag{4.5}$$

where we use the notations $B_0 = \Sigma(3)/F^2(3)$, $\bar{M}_{K,\eta}^2 = \lim_{m\to 0} M_{K,\eta}^2$. Both expressions (4.4) and (4.5) are scale independent. The connection with the SχPT $O(p^4)$ constants is $F^2(3)Z_0^S(\mu) = 16 B_0^2 L_6(\mu)$ and $F^2(3)A_0(\mu) = 16 B_0^2 L_8(\mu)$ respectively. The constant $Z_{\rm eff}^S(m_s)$, independent of $m_u = m_d = m$, is the same as in eq. (3.2), taken for $N_f = 2$. It is convenient to split the scale independent remainders $\delta M_{\pi,K}^2$ into two scale independent parts, $\delta M^2 = \delta_{(1)} M^2 + \delta_{(2)} M^2$:

$$\delta_{(1)} M_\pi^2 = \frac{4m^2 B_0^2}{32\pi^2 F_\pi^2}\left\{3\log\frac{M_K^2}{M_\pi^2} + \log\frac{M_\eta^2}{M_K^2} + \frac{m_s}{m}\left[\log\frac{\bar{M}_K^2}{M_K^2} + \frac{2}{9}\log\frac{\bar{M}_\eta^2}{M_\eta^2}\right]\right\}, \tag{4.6}$$

$$\delta_{(1)} M_K^2 = \frac{m(m + m_s) B_0^2}{32\pi^2 F_K^2}\left\{3\log\frac{M_K^2}{M_\pi^2} + \log\frac{M_\eta^2}{M_K^2} + 2\log\frac{\bar{M}_K^2}{M_K^2} + \frac{4}{9}\log\frac{\bar{M}_\eta^2}{M_\eta^2}\right\}. \tag{4.7}$$

Substituting eqs. (4.6) and (4.7) in eqs. (4.2) and (4.3), one recovers the full $O(p^4)$ standard expansion [4]. In this case, $\delta_{(2)} M^2$ consists of $O(p^6)$ (two-loop) and higher standard contributions [30]. In the GχPT reading, $\delta_{(2)} M^2$ are $O(m_{\rm quark}^3)$ as well, but now, they consist of the tree $\mathcal{L}_{(0,3)}$ component of $\mathcal{L}_{\rm eff}$, of the remaining scale independent part of the one-loop $O(p^4)$ contributions not included in eqs. (4.6) and (4.7), and of higher order terms. As a result we expect $\delta M_{\pi,K}^2 / M_{\pi,K}^2$ to be at most 3-4 per cent in the whole range $0 < X(3) < 1$ (this statement will be made more quantitative in the final result). The control of the accuracy, independently of the size of $X(3)$, is considerably simplified expanding the product $F_P^2 M_P^2$ rather than M_P^2 and F_P^2 separately. It avoids uncertainties related to the low energy constant ξ (L_5) and to its ZR violating counterpart $\tilde{\xi}$ (L_4). Further advantages of this way of ordering the expansion of Goldstone boson masses will appear shortly.

A simple algebra allows one to infer from eqs. (4.2) and (4.3) a relation between $X(3)$, the ZR violating constant $Z^S_{\rm eff}$ and the quark mass ratio $r = m_s/m$:

$$X(3) + \frac{2m(m_s+2m)}{F_\pi^2 M_\pi^2} Z^S_{\rm eff} = 1 - \tilde{\epsilon}(r) + \delta X(3)\,, \qquad (4.8)$$

where $\delta X(3)$ is a simple combination[4] of $\delta M^2_{\pi,K}$ and

$$\tilde{\epsilon}(r) = 2\frac{\tilde{r}_2 - r}{r^2-1}, \qquad \tilde{r}_2 = 2\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2} - 1 \sim 39\,. \qquad (4.9)$$

This information can now be combined with the general eq. (3.2) (considered here for $N_f = 2$). The latter can be recovered considering the limit $\Sigma(2) = \lim_{m\to 0}(F_\pi^2 M_\pi^2)/2m$ of eq. (4.2) keeping m_s fixed:

$$X(2) = X(3) + \frac{2mm_s}{F_\pi^2 M_\pi^2} Z^S_{\rm eff} + \delta X(2)\,. \qquad (4.10)$$

Eliminating $Z^S_{\rm eff}$ from eqs. (4.8) and (4.10), one arrives at a simple relation between the $N_f = 2$ and $N_f = 3$ GOR ratios (4.1) and the quark mass ratio r

$$X(2) = [1 - \tilde{\epsilon}(r)]\frac{r}{r+2} + \frac{2}{r+2}X(3) + \Delta\,. \qquad (4.11)$$

Before we show that the remainder Δ is small and well under control, the simple meaning of eq. (4.11) should be stressed. The three-flavour GOR ratio should be in the interval $0 \leq X(3) \leq X(2)$ because of the vacuum stability and of the paramagnetic inequality (2.12). Since, furthermore, $r > \tilde{r}_1 = 2F_K M_K/F_\pi M_\pi - 1 \sim 8$, the second term on the r.h.s. of eq. (4.11) represents a small correction all over the interval of r: the quark mass ratio m_s/m is merely given by the two-flavour GOR ratio $X(2)$ which is more easily accessible experimentally, whereas $X(3)$ represents the fine tuning of the above relation. It might be, for instance, conceivable that $X(2)$ would be close to 1, the quark-mass ratio r close to its standard value (or even larger) and yet $X(3) \sim 0$, implying a very steep decrease of $X(N_f)$ to the critical point and a huge amount of ZR violation. This is a possibility which has never been considered before.

We finally discuss the uncertainty in eqs. (4.8), (4.10) and (4.11). One has

$$\delta X(3) = \left[\tilde{\epsilon}(r) + \frac{2}{r-1}\right]\frac{\delta M_K^2}{M_K^2} - \frac{r+1}{r-1}\frac{\delta M_\pi^2}{M_\pi^2}\,, \qquad (4.12)$$

whereas $\delta X(2)$ can be read off from the expansion of $F_\pi^2 M_\pi^2$ inspecting linear terms in the limit $m \to 0$ with m_s fixed (including the whole $O(p^4)$ GχPT order, see

[4]Hereafter, all manipulations with eqs. (4.2) and (4.3) are algebraically exact, making no use of expansions in $\delta M^2_{\pi,K}$ or in quark masses.

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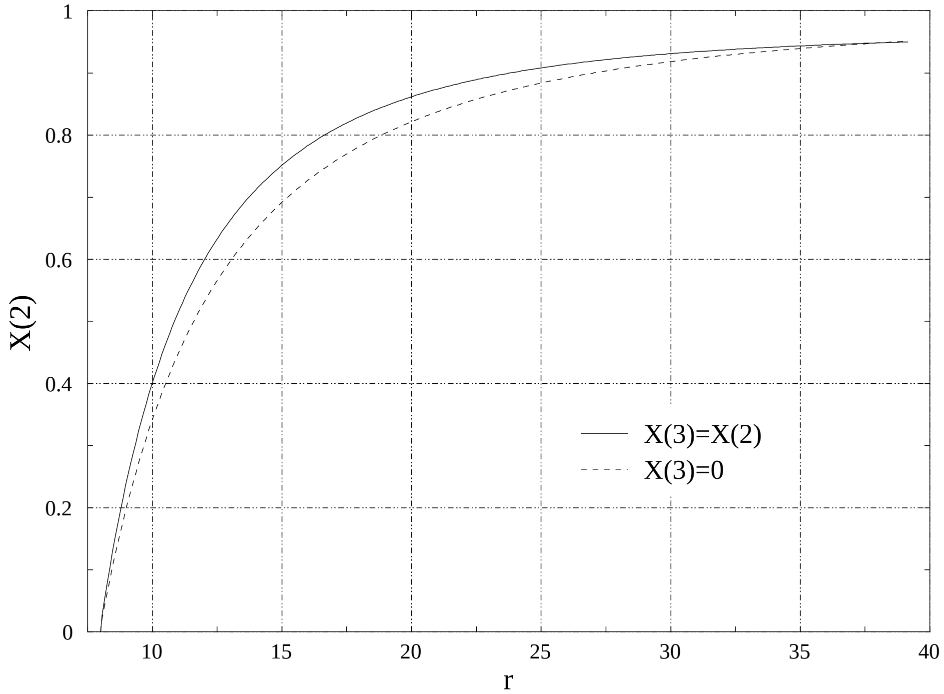

Figure 1: Correlation between the Gell-Mann-Oakes-Renner ratio for two massless flavours $X(2)$ and the quark mass ratio $r = m_s/m$. The uncertainty due to ϵ is not taken into account (see text). The upper curve respects strictly the Zweig rule, which is maximally violated on the lower curve.

eq. (5.46) in [24]). As a result, $\delta X(2) = \delta_{(2)} M_\pi^2/M_\pi^2 + \ldots$, where the dots stand for contributions for which the dimensional estimate gives $O[m^2 m_s/(M_\pi^2 \Lambda_H)] \sim \pm 0.4/r^2$. Hence, this uncertainty is at the per mille level. Δ is a simple linear combination $\Delta = \delta X(2) + \delta X(3) \cdot r/(r+2)$ and it may be represented as $\Delta = \Delta_1 \pm \epsilon$. Δ_1 represents the contributions of chiral logs to (4.12) arising from $\delta_{(1)} M_{\pi,K}^2$. The contribution of $\delta_{(2)} M_\pi^2$ to Δ almost exactly cancels between $\delta X(3)$ and $\delta X(2)$ and the remaining uncertainty ϵ is dominated by $\delta_{(2)} M_K^2$. The dimensional estimate gives $\epsilon = 0.025$ for $r = 10$, $\epsilon = 0.006$ for $r = 20$ and even much smaller values for higher r. In figure 1, the correlation between $X(2)$ and r is plotted, including Δ_1: eq. (4.10) is considered either with a maximal ZR violation $[X(3) = 0]$ or no violation at all $[X(3) = X(2)]$. We check the fine tuning role of $X(3)$ in the significant correlation between r and $X(2)$. If the two-flavour GOR ratio is close to 1, the quark mass ratio is not very accurately determined, but it is much more restricted for smaller $X(2)$.

Further insight can be obtained combining the general perturbative expressions displayed above with Moussallam's sum rule (3.6). Differentiating eq. (3.2) with respect to m_s and using eq. (4.4) yields the identity

$$X(2) - X(3) = \frac{2 m m_s}{F_\pi^2 M_\pi^2} \left[\Pi_Z(m_s) + \frac{B_0^2}{16\pi^2} \left(\bar{\lambda}_K + \frac{2}{9} \bar{\lambda}_\eta \right) \right] + \Delta X \, , \qquad (4.13)$$

where $\bar{\lambda}_P = m_s \cdot \partial(\log \bar{M}_P^2)/\partial m_s$, and

$$\Delta X = \frac{2m}{M_\pi^2 F_\pi^2} \left(1 - m_s \frac{\partial}{\partial m_s}\right) \lim_{m \to 0} \frac{F_\pi^2 \delta_{(2)} M_\pi^2}{2m} . \tag{4.14}$$

The identity (4.13) is a variant of eq. (4.10), in which the ZR violating constant $Z^S_{\rm eff}$ has not been eliminated but reexpressed using the sum rule (3.6) taken at $m = 0$. ΔX differs from $\delta X(2)$ by insertion of $(1 - m_s \partial/\partial m_s)$. They are both expected of the same order of magnitude and of opposite sign. In particular, ΔX receives contributions starting at the two-loop order in SχPT. Following the same dimensional estimate as in the discussion of the uncertainty in eq. (4.11), one expects $|\Delta X| < 0.05$ for $r \sim 10$, further decreasing for larger r.

Eq. (4.13) provides a general framework for the discussion of the variation of the condensate $X(N_f)$ between $N_f = 2$ and $N_f = 3$ in terms of the sum rule (3.6), independently of any prejudice about the size of $X(3)$. It is in particular valid in SχPT, in which the deviation of $X(3)$ from 1 is considered as a small perturbation. In the latter case, $\bar{\lambda}_K = \bar{\lambda}_\eta = 1$ at the leading order, and $2mm_s B_0^2$ can be replaced by $M_\pi^2(M_K^2 - M_\pi^2/2)$. In this way one recovers the SχPT-based analysis of ref. [11]. The second term on the right hand side of eq. (4.13) then takes the value 0.21, whereas the evaluation of the sum rule (3.6), as discussed in ref. [11], corresponds to the final result $0.38 < X(2) - X(3) < 0.73$, which is compatible with the conclusion expressed in ref. [11] in terms of the ratio $X(3)/X(2)$. Notice that $\Pi_Z(m_s) > 0$ for not too large m_s, because the two-point function (3.1) exhibits a positive logarithmic increase as $m_s \to 0$. Consequently, in SχPT, the bound $X(2) - X(3) > 0.21$ must hold up to two-loop corrections. A too large difference $X(2) - X(3)$ could hardly be reconciled with the premises of SχPT which require both GOR ratios to be reasonably close to 1. Consequently, a new analysis of the sum rule (4.13) within GχPT would be highly desirable.

5. Summary and concluding remarks

Order parameters of SBχS which are dominated by the infrared extremity of the euclidean Dirac spectrum ($\Sigma = -\langle \bar{u}u \rangle$, F_π^2, …) could be sensitive to the paramagnetic effect of light quark loops : in the chiral limit, the fermionic determinant reduces the statistical weight of the lowest Dirac eigenvalues and gradually suppresses these order parameters as the number of massless flavours N_f increases. This might result into a rich chiral phase structure as a function of N_f and N_c.

For N_f approaching the critical point $n_{\rm crit}(N_c)$, the average density of low Dirac eigenvalues should drop and its fluctuations should increase. This would naturally lead to a significant reduction of $\langle \bar{q}q \rangle$, i.e. to a rapid decrease of the Gell-Mann-Oakes-Renner (GOR) ratio $X(N_f)$ (4.1), and to an enhancement of the Zweig rule (ZR) violation in the scalar channel as compared to the large-N_c predictions.

In the real world, where $m_u \sim m_d \ll m_s \ll \Lambda_H \sim 1$ GeV, $X(2)$ and $X(3)$ are of a direct phenomenological interest ($1 > X(2) > X(3) > 0$). If the critical point (for N_c=3) is far away from $N_f = 3$, it is natural to expect $X(3) \sim X(2) \sim 1$ and a less important ZR violation in the 0^{++} channel. If, on the other hand, the real world is close to a phase transition, $X(N_f)$ should quickly fall towards the critical point, leading to a large ZR violating difference $X(2) - X(3)$. This would force a value of $X(3)$ significantly below 1, leaving open the question whether $X(2)$ already feels the influence of the critical point or still remains close to 1.

$X(2)$ can be extracted from precise low-energy $\pi\pi$-scattering experiments, independently of the ZR violation and of the size of $X(3)$. Furthermore, $X(2)$ is closely related to the quark mass ratio $r = 2m_s/(m_u + m_d)$, and this relation is only marginally affected by $X(3)$. On the other hand, even if $X(2) \sim 1$, and SχPT were a reliable expansion scheme in the two-flavour sector, its accurate predictions for s-wave scattering lengths [4, 5] would be obstructed by important ZR violation already at the one-loop level: sofar, a reliable determination of the SU(2) $\times$ SU(2) low-energy $O(p^4)$ constants l_3 and l_4 from independent experimental data requires (among other things) the knowledge of the SU(3) $\times$ SU(3) ZR violating constants L_6 and L_4. The determination of the two-flavour GOR ratio $X(2)$ remains the central goal of ongoing $\pi\pi$ scattering experiments [1, 2, 3] and of related proposals [28, 29]. If $X(2)$ turned out to be close to 1, these experiments could be interpreted as a first measurement of the low-energy constants l_3 and l_4.

Whatever the experimental output for $X(2)$ will be, additional information will be necessary to pin down $X(3)$ and to settle the theoretical issue of a nearby phase transition. The sum rule analysis of ref. [11] could be extended, but a more direct access to the ZR violation in the 0^{++} channel and to the three-flavour condensate would be desirable despite its difficulty.

The theoretical question of what happens in the phase in which we likely do not live [$N_f > n_{\mathrm{crit}}(N_c)$], is at present hard to answer unambiguously. The first chiral transition should merely affect observables that are particularly sensitive to the lowest modes of the Dirac operator. The ρ-meson mass, the string tension and the characteristics of confinement in general need not be affected at all. Among chiral order parameters, $\langle \bar{q}q \rangle$ exhibits the strongest infrared sensitivity and is expected to vanish first. Whether this implies a complete or partial restoration of chiral symmetry [31] is a question that involves the hard problem of non-perturbative regularization and renormalization of the chiral symmetry-breaking sector of QCD. Within the cut-off dependent bare theory, the vanishing of $\langle \bar{q}q \rangle$ implies $F_\pi^2 = 0$, and consequently the full restoration of chiral symmetry [32]. The validity of this argument in the full theory remains to be clarified. The answer likely resides in a non-perturbative study of the fixed points of renormalization group flows.

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